ALGEBRA FINAL EXAMINATION

This exam is of **50 marks** and is **3 hours long**. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly. Please sign the following statement:

I have not used any unfair or illegal means to answer any of the questions in this exam.

Name: **Signature:**

1. Let R be a commutative ring with 1. Let P be a *finitely generated* projective module. Show that $P^* = \mathrm{Hom}_R(P, R)$ is a projective module. 5

2. Show that an abelian group is divisible $\Leftrightarrow$ it is an injective $\mathbb{Z}$-module. 10

3. Let R be an Artinian ring. Show

3a. Every prime ideal is maximal. 5

3b. There are only finitely many prime ideals. 5

4. Let $N \in \mathbb{Z}$. What is the nilradical of the ring $\mathbb{Z}/N\mathbb{Z}$? 5

5. Let R be a commutative ring with identity. Let

$$f = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$$

be a polynomial in $R[X]$. Show that

$$f \text{ is a unit } \Leftrightarrow a_0 \text{ is a unit and } a_i \text{ are nilpotent for all } i \geq 1$$

10

6. Show that the set of zero divisors of a ring R is a union of prime ideals. (Hint: Apply Zorn's lemma to the set of ideals consisting of zero divisors.) 10